41. Let $I = \int_0^{\pi} \frac{x}{1+\cos\alpha \sin x} dx$ ………….(i)

$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)}{1+\cos\alpha \sin(\pi - x)} dx$

$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)}{1+\cos\alpha \sin x} dx$ ……….. (ii)

On adding equations (i) and (ii), we get

$$2I = \pi \int_0^{\pi} \frac{dx}{1+\cos\alpha \sin x} \quad \Rightarrow \quad 2I = \pi \int_0^{\pi} \frac{\sec^2 \frac{x}{2} dx}{\left(1+\tan^2 \frac{x}{2}\right) + 2\cos\alpha \tan\frac{x}{2}}$$

Put $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2} dx = 2dt$

$$\therefore \quad 2I = 2\pi \int_0^{\infty} \frac{dt}{1+t^2+2t\cos\alpha} \quad \Rightarrow \quad 2I = 2\pi \int_0^{\infty} \frac{dt}{(t+\cos\alpha)^2 + \sin^2\alpha}$$

$$I = \frac{\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{t+\cos\alpha}{\sin\alpha}\right)\right]_0^{\infty} = \frac{\pi}{\sin\alpha}[\tan^{-1}(\infty) - \tan^{-1}(\cot\alpha)] = \frac{\pi}{\sin\alpha}\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - \alpha\right)\right) = \frac{\alpha\pi}{\sin\alpha}$$

$$\therefore \quad I = \frac{\alpha\pi}{\sin\alpha}$$

42. Let $I = \int_0^{\pi/2} \frac{x \sin x . \cos x}{\cos^4 x + \sin^4 x} dx \quad \Rightarrow \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)\sin\left(\frac{\pi}{2} - x\right).\cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)} dx$

$$\Rightarrow \quad I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right).\sin x \cos x}{\cos^4 x + \sin^4 x} dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx - \int_0^{\pi/2} \frac{x \sin x . \cos x}{\sin^4 x + \cos^4 x} dx = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x . \cos x}{\sin^4 x + \cos^4 x} dx - I$$

$$\Rightarrow \quad 2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\tan x . \sec^2 x}{\tan^4 x + 1} dx \quad \Rightarrow \quad 2I = \frac{\pi}{2}.\frac{1}{2}\int_0^{\pi/2} \frac{1}{1+(\tan^2 x)^2} d(\tan^2 x)$$

$$\Rightarrow \quad 2I = \frac{\pi}{4}.[\tan^{-1} t]_0^{\infty} = \frac{\pi}{4}(\tan^{-1}\infty - \tan^{-1} 0) \qquad [\text{where, } t = \tan^2 x\ ]$$

$$\Rightarrow \quad I = \frac{\pi^2}{16}$$

43. Let $I = \int_0^{1/2} \frac{x \sin^{-1} x}{\sqrt{1-x^2}} dx$

Put $\sin^{-1} x = \theta \Rightarrow x = \sin\theta \quad \Rightarrow \quad dx = \cos\theta d\theta$

$\therefore \quad I = \int_0^{\pi/6} \frac{\theta \sin\theta}{\sqrt{1-\sin^2\theta}}.\cos\theta d\theta = \int_0^{\pi/6} \theta \sin\theta d\theta \;= [-\theta\cos\theta]_0^{\pi/6} + \int_0^{\pi/6} \cos\theta d\theta$

$= \left(-\frac{\pi}{6}\cos\frac{\pi}{6} + 0\right) + \left(\sin\frac{\pi}{6} - \sin 0\right) = -\frac{\sqrt{3}\pi}{12} + \frac{1}{2}$

44. Let $I = \int_0^{\pi/4} \frac{(\sin x + \cos x)}{9 + 16\sin 2x} dx$

$I = \int_0^{\pi/4} \frac{\sin x + \cos x}{25 - 16\,(\sin x - \cos x)^2} dx$

Put $4(\sin x - \cos x) = t \quad \Rightarrow \quad 4(\cos x + \sin x)dx = dt$

$\therefore \quad I = \frac{1}{4}\int_{-4}^{0} \frac{dt}{25 - t^2} = \frac{1}{4}.\frac{1}{2(5)} \log\left[\left|\frac{5+t}{5-t}\right|\right]_{-4}^{0}$

$= \frac{1}{40}\left[\log\left|\frac{5+0}{5-0}\right| - \log\left|\frac{5-4}{5+4}\right|\right]$

$= \frac{1}{40}\left(\log 1 - \log\frac{1}{9}\right) = \frac{1}{40}\log 9 = \frac{1}{20}(\log 3)$

45. (i) Let $I = \int_0^{\pi} x \; f(\sin x)\, dx$ …………(i)

$\Rightarrow \quad I = \int_0^{\pi} (\pi - x)\, f(\sin x)\, dx$ …………(ii)

On adding equation (i) and (ii), we get

$2I = \int_0^{\pi} \pi f(\sin x)\, dx$

$\therefore \quad \int_0^{\pi} x \; f(\sin x) dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x) dx$

(ii) Let $\quad I = \int_{-1}^{3/2} |\, x \sin \pi x \,|\, dx$

Since, $|\, x\sin \pi x \,| = \begin{cases} x\sin\pi x, & -1 < x \le 1 \\ -x\sin\pi x, & 1 < x < \frac{3}{2} \end{cases}$

$\therefore \quad I = \int_{-1}^{1} x\sin\pi x\, dx + \int_{1}^{3/2} -x\sin\pi x\, dx$

$= 2\left[-\frac{x\cos\pi}{\pi}\right]_0^1 - 2\int_0^1 1.\left(\frac{-\cos\pi x}{\pi}\right)dx - \left\{\left[\frac{-\cos\pi x}{\pi}\right]_1^{3/2} - \int_1^{3/2}\left(\frac{-\cos\pi x}{\pi}\right)dx\right\}$

$= 2\left(\frac{1}{\pi}\right) + \frac{2}{\pi}.\left[\frac{\sin\pi x}{\pi}\right]_0^1 - \left(-\frac{1}{\pi}\right) - \frac{1}{\pi}\left[\frac{\sin\pi x}{\pi}\right]_1^{3/2}$

$= \frac{2}{\pi} + \frac{2}{\pi^2}(0-0) + \frac{1}{\pi} + \frac{1}{\pi^2}(+1-0) = \frac{3}{\pi} + \frac{1}{\pi^2} = \left(\frac{3\pi+1}{\pi^2}\right)$

46. Let $I = \int_0^1 (tx+1-x)^n dx = \int_0^1 \{(t-1)x+1\}^n dx = \left[\frac{((t-1)x+1)^{n+1}}{(n+1)(t-1)}\right]_0^1 = \frac{1}{n+1}\left(\frac{t^{n+1}-1}{t-1}\right)$

$= \frac{1}{n+1}(1+t+t^2+....+t^n)$ ....(i)

Again, $I = \int_0^1 (tx+1-x)^n dx = \int_0^1 [(1-x)+tx]^n\ dx$

$I = \int_0^1 [{}^nC_0(1-x)^n + {}^nC_1(1-x)^{n-1}(tx) + {}^nC_2(1-x)^{n-2}(tx)^2 + ... + {}^nC_n(tx)^n]dx$

$= \int_0^1 \left[\sum_{r=0}^{n} {}^nC_r(1-x)^{n-r}(tx)^r\right]dx = \sum_{r=0}^{n} {}^nC_r\left[\int_0^1 (1-x)^{n-r}.x^r dx\right]t^r$ ....(ii)

From equation (i) and (ii), we get

$\sum_{r=0}^{n} {}^nC_r\left[\int_0^1 (1-x)^{n-r}.x^r dx\right]t^r = \frac{1}{n+1}(1+t+....+t^n)$

On equating coefficient of t^k on both sides, we get

${}^nC_k\left[\int_0^1 (1-x)^{n-k}.x^k dx\right] = \frac{1}{n+1} \Rightarrow \int_0^1 (1-x)^{n-k}x^k dx = \frac{1}{(n+1)\,{}^nC_k}$

47.(2) Given, $I = \int_0^1 4x^3 \frac{d^2}{dx^2}(1-x^2)^5 dx = \left[4x^3 \frac{d}{dx}(1-x^2)^5\right] - \int_0^1 12x^2 \frac{d}{dx}(1-x^2)^5 dx$

$= \left[4x^3 \times 5(1-x^2)^4(-2x)\right]_0^1$

$-12\left[[x^2(1-x^2)^5]_0^1 - \int_0^1 2x(1-x^2)^5 dx\right] = 0 - 0 - 12(0-0) + 12\int_0^1 2x(1-x^2)^5 dx$

$= 12 \times \left[-\frac{(1-x^2)^6}{6}\right] = 12\left[0+\frac{1}{6}\right] = 2$

48.(B) Given, $g(x) = \int_0^x f(t)dt$

$g(2) = \int_0^2 f(t)dt = \int_0^1 f(t)dt + \int_1^2 f(t)dt$

Now, $\frac{1}{2} \le f(t) \le 1$ for $t \in [0,1]$

We get $\int_0^1 \frac{1}{2}dt \le \int_0^1 f(t)dt \le \int_0^1 1dt$

$\Rightarrow \quad \frac{1}{2} \le \int_0^1 f(t)dt \le 1$ ....(i)

Again, $0 \le f(t) \le \frac{1}{2}$ for $t \in [1, 2]$ ....(ii)

$\Rightarrow \quad \int_1^2 0\,dt \le \int_1^2 f(t)dt \le \int_1^2 \frac{1}{2}dt \Rightarrow 0 \le \int_1^2 f(t)dt \le \frac{1}{2}$

From equation (i) and (ii), we get

$\frac{1}{2} \le \int_0^1 f(t)dt + \int_1^2 f(t)dt \le \frac{3}{2} \quad \Rightarrow \quad \frac{1}{2} \le g(2) \le \frac{3}{2} \Rightarrow 0 \le g(2) < 2$

49. $\int_0^{n\pi+v} |\sin x|\, dx = \int_0^{\pi} |\sin x|\, dx + \int_{\pi}^{2\pi} |\sin x|\, dx + \ldots + \int_{(n-1)\pi}^{n\pi} |\sin x|\, dx + \int_{n\pi}^{n\pi+v} |\sin x|\, dx$

$\sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} |\sin x|\, dx + \int_{n\pi}^{n\pi+v} |\sin x|\, dx$

Now to solve, $\int_{(r-1)\pi}^{r\pi} |\sin x|\, dx$, we have $x = (r-1)\pi + t$

$\therefore \quad \int_{(r-1)\pi}^{r\pi} |\sin x| dx = \int_0^{\pi} |(-1)^{r-1} \sin t|\, dt$

$= \int_0^{\pi} |\sin t|\, dt = \int_0^{\pi} |\sin t|\, dt \quad = [-\cos t]_0^{\pi} = -\cos\pi + \cos 0 = 2$

Again, $\int_{n\pi}^{n\pi+v} |\sin x|\, dx$, putting $x = n\pi + t$

Then, $\int_{n\pi}^{n\pi+v} |\sin x|\, dx = \int_0^{v} |(-1)^n \sin t| dt = \int_0^{v} \sin t\, dt$

$= [-\cos t]_0^{v} = -\cos v + \cos 0 = 1 - \cos v$

$\therefore \quad \int_0^{n\pi+v} |\sin x| dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} |\sin x|\, dx + \int_{n\pi}^{n\pi+v} |\sin x| dx$

$\sum_{r=1}^{n} 2 + \int_{n\pi}^{n\pi+v} |\sin x| dx = 2n + 1 - \cos v$

50. Let $\phi(a) = \int_a^{a+t} f(x) dx$

On differentiating w.r.t a, we get

$\phi'(a) = f(a+t)\cdot 1 - f(a)\cdot 1 = 0 \qquad$ [given, $f(x+t) = f(x)$]

$\therefore \quad \phi(a)$ is constant.

$\Rightarrow \quad \int_a^{a+t} f(x)\, dx$ is independent of a